B.sc(H)) part 1 paper 1
Topic:Resolution into Factors
Subject:Mathematics
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Theorem

Resolution of sin θ into Factors : Schlomilch Method

We have, $\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = 2\sin\frac{x}{2}\sin\frac{\pi+x}{2}$ ...(1)

Then $\sin\theta = 2\sin\frac{\theta}{2}\sin\frac{\pi+\theta}{2}$ ...(2)

Putting $\frac{\theta}{2}$ and $\frac{\pi+\theta}{2}$ for x in (1) successively, we have

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}$$

and
$$\sin\frac{\pi+\theta}{2} = 2\sin\frac{\pi+\theta}{2^2}\cdot\sin\frac{3\pi+\theta}{2^2}$$

Substituting these values in the R.H.S. of (2), and rearranging we have

$$\sin\theta = 2^3\sin\frac{\theta}{2^2}\cdot\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

Continuing this process we have

$$\sin\theta = 2^7\sin\frac{\theta}{2^3}\cdot\sin\frac{\pi+\theta}{2^3}\cdot\sin\frac{2\pi+\theta}{2^3}\cdot\sin\frac{3\pi+\theta}{2^3}\ldots\sin\frac{7\pi+\theta}{2^3}$$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$= 2^{2^n-1}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\cdot\sin\frac{2\pi+\theta}{2^n}\ldots$$
$$\ldots\sin\frac{(2^n-2)\pi+\theta}{2^n}\cdot\sin\frac{(2^n-1)\pi+\theta}{2^n}$$

$$= 2^{p-1} \sin\frac{\theta}{p} \sin\frac{\pi+\theta}{p} \cdot \sin\frac{2\pi+\theta}{p} \ldots$$

$$\ldots \sin\frac{(p-2)\pi+\theta}{p} \cdot \sin\frac{(p-1)\pi+\theta}{p} \quad \ldots(3)$$

where $p = 2^n$.

The last factor in (3), viz;

$$\sin\frac{(p-1)\pi+\theta}{p} = \sin\left(\pi - \frac{\pi-\theta}{p}\right) = \sin\frac{\pi-\theta}{p}$$

The last factor but one

$$\sin\frac{(p-2)\pi+\theta}{p} = \sin\left(\pi - \frac{2\pi-\theta}{p}\right) = \sin\frac{2\pi-\theta}{p}$$

and so on.

Now combine together, the second and the last factors, the third and last but one and so on.

Since p is an even number, the number of factors in (3) is even; thus leaving the first factor alone, the number of factors thus remained, would be $(p-1)$ i.e. an odd number, which we have to combine according to the scheme above. The middle factor would thus be $\frac{p}{2}$th factor as counted from the second factor or the $\left(\frac{p}{2}+1\right)$th factor from the first which cannot be grouped with any factor, and its value is

$$\sin\frac{\frac{p}{2}\pi+\theta}{p} = \sin\left(\frac{\pi}{2}+\frac{\theta}{p}\right) = \cos\frac{\theta}{p}$$

The equation (3) thus becomes

$$\sin\theta = 2^{p-1} \cdot \sin\frac{\theta}{p}\left\{\sin^2\frac{\pi}{p} - \sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p} - \sin^2\frac{\theta}{p}\right\}$$

$$\ldots\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} - \sin^2\frac{\theta}{p}\right\}\cos\frac{\theta}{p} \quad \ldots(4)$$

Divide both sides by $\sin\frac{\theta}{p}$ and let $\theta \to 0$, then we have

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1; \quad \underset{\theta \to 0}{\text{Lt}}\left(\sin^2\frac{\theta}{p}\right) = 0$$

and $\underset{\theta \to 0}{\text{Lt}} \left(\dfrac{\sin \theta}{\sin \dfrac{\theta}{p}} \right) = \underset{\theta \to 0}{\text{Lt}} \left(\dfrac{\sin \theta}{\theta} \right) \left(\dfrac{\dfrac{\theta}{p}}{\sin \dfrac{\theta}{p}} \right) \cdot p = p$

Thus (4) gives $p = 2^{p-1} \sin^2 \dfrac{\pi}{p} \sin^2 \dfrac{2\pi}{p} \ldots \sin^2 \dfrac{\left(\dfrac{p}{2} - 1\right)\pi}{p}$ ...(5)

Dividing (4) by (5), we get

$$\sin \theta = p \sin \frac{\theta}{p} \left\{ 1 - \frac{\sin^2 \dfrac{\theta}{p}}{\sin^2 \dfrac{\pi}{p}} \right\} \left\{ 1 - \frac{\sin^2 \dfrac{\theta}{p}}{\sin^2 \dfrac{2\pi}{p}} \right\} \ldots$$

$$\ldots \left\{ 1 - \frac{\sin^2 \dfrac{\theta}{p}}{\sin^2 \dfrac{\left(\dfrac{p}{2} - 1\right)\pi}{p}} \right\} \cos \frac{\theta}{p} \quad \ldots(6)$$

Now let $p \to \infty$, so that

$$\underset{p \to \infty}{\text{Lt}} \left(p \sin \frac{\theta}{p} \right) = \underset{p \to \infty}{\text{Lt}} \left(\frac{\sin \dfrac{\theta}{p}}{\dfrac{\theta}{p}} \right) \cdot \theta = \theta,$$

and $$\underset{p \to \infty}{\text{Lt}} \left(\frac{\sin^2 \dfrac{\theta}{p}}{\sin^2 \dfrac{r\pi}{p}} \right) = \underset{p \to \infty}{\text{Lt}} \left\{ \frac{\sin^2 \dfrac{\theta}{p}}{\left(\dfrac{\theta}{p}\right)^2} \cdot \frac{\left(\dfrac{r\pi}{p}\right)^2}{\sin^2 \dfrac{r\pi}{p}} \cdot \frac{\theta^2}{r^2 \pi^2} \right\}$$

$$= \frac{\theta^2}{r^2 \pi^2}.$$

Thus we have from the above equation

$$\sin\theta = \theta \left(1 - \frac{\theta^2}{\pi^2} \right) \left(1 - \frac{\theta^2}{2^2 \pi^2} \right) \left(1 - \frac{\theta^2}{3^2 \pi^2} \right)$$

$$= \theta \prod_{r=1}^{\infty} \left(1 - \frac{\theta^2}{r^2 \pi^2} \right)$$